

# Entropy Bounds for Discrete Random Variables via Coupling

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**Abstract**—This paper derives new entropy bounds for discrete random variables via maximal coupling. It provides bounds on the difference between the entropies of two discrete random variables in terms of the local and total variation distances between their probability mass functions. These bounds address cases of finite or countable infinite alphabets. Particular cases of these bounds reproduce some known results. The use of the new entropy bounds is exemplified by relying on some bounds on the above distances via Stein's method. The improvement that is obtained by these bounds is exemplified.

**Index Terms**—Coupling, entropy, local distance, Stein's method, total variation distance.

## I. INTRODUCTION

Inequalities that relate the Shannon entropy or information divergence with the total variation distance were extensively studied during the last fifty years (see, e.g., [7]–[10], [13], [14], [16], [18]–[21], [25]–[27], [30]–[38], [42], [44]–[49]). Among the observations in these works, it is known that a sufficiently small total variation distance between a pair of discrete random variables with a *finite and fixed alphabet*, implies a small difference between their entropies. However, if the size of the alphabet is finite but it is not bounded then for an arbitrarily small  $\delta > 0$  and an arbitrarily large  $\mu > 0$ , there exists a pair of discrete random variables such that the total variation distance between them is less than  $\delta$  whereas the difference between their entropies is larger than  $\mu$  (see, e.g., [21, Theorem 1] with a concrete example in its proof).

The interplay between the entropy difference of two discrete random variables and their total variation distance was studied in [7, Theorem 17.3.3] or [10, Lemma 2.7], [11, Lemma 1], [21], [34], [42, Section 2] and [49]. The bounds that are derived in this work improve some existing bounds as a result of their dependence on both the *local and total variation distances* and the alphabet sizes (the relevant distances are defined later in this section). The new bounds are derived via the use of *maximal coupling*, which is also known to be useful for the derivation of error bounds via Stein's method (see, e.g., [40, Chapter 2] and [41]). It is noted that the entropy bounds in [49] are also derived via coupling, but the approach of the analysis in this work is remarkably different (see Sections II and III). The new bounds are linked to Stein's method, and the improvement that is achieved by these bounds is exemplified.

We provide in the following the essential mathematical background that is required for the analysis in this work.

**Definition 1:** A *coupling* of a pair of two discrete random variables  $(X, Y)$  is a pair of two random variables  $(\hat{X}, \hat{Y})$  such that the marginal distributions of  $(X, Y)$  and  $(\hat{X}, \hat{Y})$  coincide, i.e.,  $P_X = P_{\hat{X}}$  and  $P_Y = P_{\hat{Y}}$ .

**Definition 2:** For a pair of random variables  $(X, Y)$ , a coupling  $(\hat{X}, \hat{Y})$  is called a *maximal coupling* if  $\mathbb{P}(\hat{X} = \hat{Y})$  is as large as possible among all the couplings of  $(X, Y)$ .

The following theorem is a basic result on maximal coupling that also suggests, as part of its proof, a construction for maximal coupling. We later rely on this particular construction to derive in Section III some new bounds on the entropy of discrete random variables. Hence, the proof of the following known theorem serves for the analysis in this work.

**Theorem 1:** Let  $X$  and  $Y$  be discrete random variables that take values in a set  $\mathcal{A}$ , and let their respective probability mass functions be

$$P_X(x) = \mathbb{P}(X = x), \quad P_Y(y) = \mathbb{P}(Y = y), \quad \forall x, y \in \mathcal{A}.$$

Then, the maximal coupling of  $(X, Y)$  satisfies

$$\mathbb{P}(\hat{X} = \hat{Y}) = \sum_{u \in \mathcal{A}} \min\{P_X(u), P_Y(u)\}. \quad (1)$$

**Proof:** Let  $\mathcal{B} \triangleq \{u \in \mathcal{A} : P_X(u) < P_Y(u)\}$ , and let  $\mathcal{B}^c \triangleq \mathcal{A} \setminus \mathcal{B}$ . Then, for every coupling  $(\hat{X}, \hat{Y})$  of  $(X, Y)$ ,

$$\begin{aligned} \mathbb{P}(\hat{X} = \hat{Y}) &= \mathbb{P}(\hat{X} = \hat{Y} \in \mathcal{B}) + \mathbb{P}(\hat{X} = \hat{Y} \in \mathcal{B}^c) \\ &\leq \mathbb{P}(\hat{X} \in \mathcal{B}) + \mathbb{P}(\hat{Y} \in \mathcal{B}^c) \\ &= \mathbb{P}(X \in \mathcal{B}) + \mathbb{P}(Y \in \mathcal{B}^c) \\ &= \sum_{u \in \mathcal{B}} P_X(u) + \sum_{u \in \mathcal{B}^c} P_Y(u) \\ &= \sum_{u \in \mathcal{B}} \min\{P_X(u), P_Y(u)\} + \sum_{u \in \mathcal{B}^c} \min\{P_X(u), P_Y(u)\} \\ &= \sum_{u \in \mathcal{A}} \min\{P_X(u), P_Y(u)\} \triangleq p. \end{aligned} \quad (2)$$

The following provides a construction of a coupling  $(\hat{X}, \hat{Y})$  that achieves the bound in (2) with equality, so it forms a maximal coupling of  $(X, Y)$ . Let  $U$ ,  $V$ ,  $W$  and  $J$  be independent discrete random variables, where

$$\mathbb{P}(J = 0) = 1 - p, \quad \mathbb{P}(J = 1) = p \quad (3)$$

so  $J \sim \text{Bernoulli}(p)$ , and let  $U$ ,  $V$ ,  $W$  have the following probability mass functions:

$$P_U(u) = \frac{\min\{P_X(u), P_Y(u)\}}{p}, \quad \forall u \in \mathcal{A} \quad (4)$$

$$P_V(v) = \frac{P_X(v) - \min\{P_X(v), P_Y(v)\}}{1 - p}, \quad \forall v \in \mathcal{A} \quad (5)$$

$$P_W(w) = \frac{P_Y(w) - \min\{P_X(w), P_Y(w)\}}{1 - p}, \quad \forall w \in \mathcal{A}. \quad (6)$$

If  $J = 1$ , let  $\hat{X} = \hat{Y} = U$ , and if  $J = 0$  let  $\hat{X} = V$  and  $\hat{Y} = W$ . For every  $x, y \in \mathcal{A}$

$$\begin{aligned} P_{\hat{X}}(x) &= p \mathbb{P}(\hat{X} = x | J = 1) + (1 - p) \mathbb{P}(\hat{X} = x | J = 0) \\ &= p P_U(x) + (1 - p) P_V(x) \\ &= P_X(x) \end{aligned}$$

and similarly  $P_{\hat{Y}}(y) = P_Y(y)$ , so  $(\hat{X}, \hat{Y})$  is indeed a coupling of  $(X, Y)$ . Furthermore,

$$\mathbb{P}(\hat{X} = \hat{Y}) \geq \mathbb{P}(J = 1) = p \quad (7)$$

so, from (2) and (7), it follows that the proposed construction for  $(\hat{X}, \hat{Y})$  is a maximal coupling of  $(X, Y)$ . ■

*Definition 3:* Let  $X$  and  $Y$  be discrete random variables that take values in a set  $\mathcal{A}$ , and let  $P_X$  and  $P_Y$  be their respective probability mass functions. The *local distance* and *total variation distance* between  $X$  and  $Y$  are, respectively,

$$d_{\text{loc}}(X, Y) \triangleq \sup_{u \in \mathcal{A}} |P_X(u) - P_Y(u)| \quad (8)$$

$$d_{\text{TV}}(X, Y) \triangleq \frac{1}{2} \sum_{u \in \mathcal{A}} |P_X(u) - P_Y(u)|. \quad (9)$$

Without abuse of notation, one can also write  $d_{\text{loc}}(P_X, P_Y)$  and  $d_{\text{TV}}(P_X, P_Y)$ , respectively.

*Remark 1:* The factor of one-half on the right-hand side of (9) normalizes the total variation distance to get values between zero and one. It is noted that the notation in the literature is not consistent, with a factor 2 on the right-hand side of (9) often being present or not. It is easy to show (see, e.g., [16, Lemma 5.4 on pp. 133–134]) that

$$d_{\text{TV}}(X, Y) = \sup_{\mathcal{B} \subseteq \mathcal{A}} |\mathbb{P}(X \in \mathcal{B}) - \mathbb{P}(Y \in \mathcal{B})|.$$

From the last equality and the definition of the local distance in (8), it follows that  $d_{\text{loc}}(X, Y) \leq d_{\text{TV}}(X, Y)$ .

The following result is a simple consequence of Theorem 1, and it is also used for the derivation of the new bounds on the entropy in Section III.

*Theorem 2:* Let  $X$  and  $Y$  be two discrete random variables that take values in a set  $\mathcal{A}$ . If  $(\hat{X}, \hat{Y})$  is a maximal coupling of  $(X, Y)$  then

$$\mathbb{P}(\hat{X} \neq \hat{Y}) = d_{\text{TV}}(X, Y). \quad (10)$$

*Proof:* This follows from (1) and (9), and the equality  $\min\{a, b\} = \frac{a+b-|a-b|}{2}$  for  $a, b \in \mathbb{R}$ . ■

The continuation of this paper is structured as follows: Section II provides a simple proof, via maximal coupling, for an existing bound on the difference between the entropies of two discrete random variables in terms of their total variation distance (see [21, Theorem 6] and [49, Eq. (4)]). The proof of this bound is a shortened version of the proof in [49], and it serves to motivate the derivation of some refined bounds on the difference between the entropies of two discrete random variables. These new bounds, proved in Section III via maximal coupling, depend on both the local and total variation distances. Section IV exemplifies the use of the new bounds with a link to Stein's method, and it also compares them with some existing bounds.

## II. A PROOF OF A KNOWN BOUND ON THE ENTROPY OF DISCRETE RANDOM VARIABLES VIA COUPLING

The following theorem relies on a bound that first appeared in [49, Eq. (4)] and proved by coupling. It was later introduced in [21, Theorem 6] by re-proving the inequality in a different way (without coupling), and it was also strengthened there by showing an explicit case where the following bound is tight. As is proved in [49, Section 3], the bound on the entropy difference that is introduced in the following theorem improves the bound in [7, Theorem 17.3.3] or [10, Lemma 2.7].

*Theorem 3:* Let  $X$  and  $Y$  be two discrete random variables that take values in a set  $\mathcal{A}$ , and let  $|\mathcal{A}| = M$ . If  $d_{\text{TV}}(X, Y) \leq \varepsilon$ , then

$$|H(X) - H(Y)| \leq \begin{cases} \varepsilon \log(M-1) + h(\varepsilon) & \text{if } \varepsilon \in [0, 1 - \frac{1}{M}] \\ \log(M) & \text{if } \varepsilon > 1 - \frac{1}{M} \end{cases}$$

where  $h$  denotes the binary entropy function. Furthermore, there is a case where the bound is achieved with equality.

The following proof of Theorem 3 exemplifies the use of maximal coupling in proving an information-theoretic result.

*Proof:* Let  $(\hat{X}, \hat{Y})$  be a maximal coupling of  $(X, Y)$ . Since  $H(X) = H(\hat{X})$  and  $H(Y) = H(\hat{Y})$  (note that the marginal probability mass functions of  $(X, Y)$  and  $(\hat{X}, \hat{Y})$  are the same), it follows from Fano's inequality and Theorem 2 (see (10)) that

$$\begin{aligned} |H(X) - H(Y)| &= |H(\hat{X}) - H(\hat{Y})| \\ &= |H(\hat{X}|\hat{Y}) - H(\hat{Y}|\hat{X})| \\ &\leq \max\{H(\hat{X}|\hat{Y}), H(\hat{Y}|\hat{X})\} \\ &\leq \mathbb{P}(\hat{X} \neq \hat{Y}) \log(M-1) + h(\mathbb{P}(\hat{X} \neq \hat{Y})) \\ &= d_{\text{TV}}(X, Y) \log(M-1) + h(d_{\text{TV}}(X, Y)). \end{aligned}$$

This proves the bound in [49, Eq. (4)]. If  $d_{\text{TV}}(X, Y) \leq \varepsilon$  for some  $\varepsilon \in [0, 1 - \frac{1}{M}]$ , the replacement of  $d_{\text{TV}}(X, Y)$  in the last bound by  $\varepsilon$  is valid; this holds since the function  $f(x) \triangleq x \log(M-1) + h(x)$  is monotonic increasing over the interval  $[0, 1 - \frac{1}{M}]$  (since  $f'(x) = \log(M-1) + \log(\frac{1-x}{x}) > 0$  for  $0 < x < 1 - \frac{1}{M}$ ). Otherwise, if  $\varepsilon > 1 - \frac{1}{M}$ , then

$$|H(X) - H(Y)| \leq \max\{H(X), H(Y)\} \leq \log(M).$$

*Cases where the bound is tight [21]:* If  $\varepsilon \in [0, 1 - \frac{1}{M}]$ , the bound is tight when

$$\begin{aligned} X &\sim P_X = \left(1 - \varepsilon, \frac{\varepsilon}{M-1}, \dots, \frac{\varepsilon}{M-1}\right) \\ Y &\sim P_Y = (1, 0, \dots, 0) \end{aligned}$$

which implies that

$$\begin{aligned} d_{\text{TV}}(X, Y) &= \varepsilon, \\ |H(X) - H(Y)| &= H(X) = h(\varepsilon) + \varepsilon \log(M-1). \end{aligned}$$

If  $\varepsilon \in (1 - \frac{1}{M}, 1]$  then the bound is tight when

$$X \sim \left(\frac{1}{M}, \dots, \frac{1}{M}\right), \quad Y \sim (1, 0, \dots, 0)$$

so,  $d_{\text{TV}}(X, Y) = 1 - \frac{1}{M} < \varepsilon$  and  $|H(X) - H(Y)| = \log(M)$ . ■

### III. NEW BOUNDS ON THE ENTROPY OF DISCRETE RANDOM VARIABLES VIA COUPLING

In the cases where the known bound in Theorem 3 was shown to be tight in [21] (see the last part of the proof in Section II), it is easy to verify that the local distance is equal to the total variation distance. However, as is shown in the following, if it is not the case (i.e., the local distance is smaller than the total variation distance), then the bound in Theorem 3 is *necessarily* not tight. Furthermore, this section provides new bounds that depend on both the total variation and local distances. If these two distances are equal then the new bound is particularized to the bound in Theorem 3 but otherwise, the new bound improves the bound in Theorem 3. The general approach for proving the following new inequalities relies on the construction of the maximal coupling that is introduced in the proof of Theorem 1. The new results are stated and proved in the following.

**Theorem 4:** Let  $X$  and  $Y$  be two discrete random variables that take values in a set  $\mathcal{A}$ , and let  $|\mathcal{A}| = M$ . Then,

$$|H(X) - H(Y)| \leq d_{\text{TV}}(X, Y) \log(M\alpha - 1) + h(d_{\text{TV}}(X, Y)) \quad (11)$$

where

$$\alpha \triangleq \frac{d_{\text{loc}}(X, Y)}{d_{\text{TV}}(X, Y)} \quad (12)$$

denotes the ratio of the local and total variation distances (so,  $\alpha \in [\frac{2}{M}, 1]$ ), and  $h$  denotes the binary entropy function. Furthermore, if the probability mass functions of  $X$  and  $Y$  satisfy the condition that  $\frac{1}{2} \leq \frac{P_X}{P_Y} \leq 2$  whenever  $P_X, P_Y > 0$ , then the bound in (11) is tightened to

$$|H(X) - H(Y)| \leq d_{\text{TV}}(X, Y) \log\left(\frac{M\alpha - 1}{4}\right) + h(d_{\text{TV}}(X, Y)). \quad (13)$$

**Remark 2:** Since, in general,  $\alpha \leq 1$  then the case where  $\alpha = 1$  is the worst case for the bound in (11). In the latter case, it is particularized to the bound in Theorem 3 (see [21, Theorem 6] or [49, Eq. (4)]).

**Remark 3:** If  $\alpha \leq \frac{1}{N}$  for some integer  $N$  (since  $\alpha \in [\frac{2}{M}, 1]$  then  $N \in \{1, \dots, \lfloor \frac{M}{2} \rfloor\}$ ), the bound in (11) implies that

$$|H(X) - H(Y)| \leq d_{\text{TV}}(X, Y) \log\left(\frac{M - N}{N}\right) + h(d_{\text{TV}}(X, Y)). \quad (14)$$

The bounds in (14) and [21, Theorem 7] are similar *but they hold under different conditions*. The bound in [21, Theorem 7] requires that  $P_X, P_Y \leq \frac{1}{N}$  everywhere, whereas the bound in (14) holds under the requirement that the ratio  $\alpha$  of the local and total variation distances satisfies  $\alpha \leq \frac{1}{N}$ . None of these conditions implies the other.

We prove in the following Theorem 4.

**Proof:** Assume without loss of generality (w.o.l.o.g.) that  $H(X) - H(Y) \geq 0$  (note that there is a symmetry between  $X$  and  $Y$  in  $|H(X) - H(Y)|$ ,  $d_{\text{loc}}(X, Y)$  and  $d_{\text{TV}}(X, Y)$ ).

Let  $(\hat{X}, \hat{Y})$  be the maximal coupling of  $(X, Y)$  according to the construction in the proof of Theorem 1. Then,

$$\begin{aligned} |H(X) - H(Y)| &= H(X) - H(Y) \\ &= H(\hat{X}) - H(\hat{Y}) \\ &= H(\hat{X}|J) - H(\hat{Y}|J) + I(\hat{X}; J) - I(\hat{Y}; J). \end{aligned} \quad (15)$$

The conditional entropy  $H(\hat{X}|J)$  satisfies

$$\begin{aligned} H(\hat{X}|J) &= \mathbb{P}(J = 0) H(\hat{X}|J = 0) + \mathbb{P}(J = 1) H(\hat{X}|J = 1) \\ &\stackrel{(a)}{=} d_{\text{TV}}(X, Y) H(V|J = 0) + (1 - d_{\text{TV}}(X, Y)) H(U|J = 1) \\ &\stackrel{(b)}{=} d_{\text{TV}}(X, Y) H(V) + (1 - d_{\text{TV}}(X, Y)) H(U) \end{aligned} \quad (16)$$

where equality (a) holds since  $J \sim \text{Bernoulli}(p)$  with

$$p = \mathbb{P}(J = 1) = \mathbb{P}(\hat{X} = \hat{Y}) = 1 - d_{\text{TV}}(X, Y)$$

(see the proof of Theorem 1 and the result in Theorem 2), and because  $\hat{X}$  is equal to  $V$  or  $U$  when  $J$  gets that values zero or one, respectively. Furthermore, equality (b) holds since  $U, V, W, J$  are independent random variables (due to the construction shown in the proof of Theorem 1). Similarly,

$$H(\hat{Y}|J) = d_{\text{TV}}(X, Y) H(W) + (1 - d_{\text{TV}}(X, Y)) H(U). \quad (17)$$

Combining (15)–(17) yields that

$$\begin{aligned} |H(X) - H(Y)| &= d_{\text{TV}}(X, Y) (H(V) - H(W)) + I(\hat{X}; J) - I(\hat{Y}; J). \end{aligned} \quad (18)$$

From (5) and (6), it follows that

$$P_V(a) P_W(a) = 0, \quad \forall a \in \mathcal{A} \quad (19)$$

and also, for every  $a \in \mathcal{A}$ ,

$$\begin{aligned} P_V(a) + P_W(a) &= \frac{P_X(a) + P_Y(a) - 2 \min\{P_X(a), P_Y(a)\}}{d_{\text{TV}}(X, Y)} \\ &= \frac{|P_X(a) - P_Y(a)|}{d_{\text{TV}}(X, Y)} \\ &\leq \frac{d_{\text{loc}}(X, Y)}{d_{\text{TV}}(X, Y)} \triangleq \alpha. \end{aligned} \quad (20)$$

In the following, we derive upper bounds on  $H(V) - H(W)$  and  $I(\hat{X}; J) - I(\hat{Y}; J)$ , and rely on (18) to get an upper bound on  $|H(X) - H(Y)|$ . Let  $\mathcal{A} \triangleq \{a_1, \dots, a_M\}$ , and

$$s_i \triangleq P_V(a_i), \quad t_i \triangleq P_W(a_i), \quad \forall i \in \{1, \dots, M\}.$$

From (19) and (20),

$$s_i t_i = 0, \quad s_i + t_i \leq \alpha, \quad \forall i \in \{1, \dots, M\}$$

and  $H(V) - H(W) = -\sum_{i=1}^M s_i \log(s_i) + \sum_{i=1}^M t_i \log(t_i)$ . Hence, for fixed  $\alpha$  and  $M$  (since  $|\mathcal{A}| = M$ , then  $\alpha \in [\frac{2}{M}, 1]$ ),

$$H(V) - H(W) \leq g(\alpha) \quad (21)$$

where  $g(\alpha)$  is the solution of the optimization problem

$$\begin{aligned} & \text{maximize} \left( -\sum_{i=1}^M s_i \log(s_i) + \sum_{i=1}^M t_i \log(t_i) \right) \\ & \text{subject to} \\ & \begin{cases} s_i, t_i \geq 0, \quad s_i + t_i \leq \alpha \\ s_i t_i = 0, \quad \forall i \in \{1, \dots, M\} \\ \sum_{i=1}^M s_i = \sum_{i=1}^M t_i = 1 \end{cases} \end{aligned} \quad (22)$$

with the  $2M$  variables  $s_1, t_1, \dots, s_M, t_M$ . Fortunately, this non-convex optimization problem admits a closed-form solution.

*Lemma 1:* The solution of the non-convex optimization problem in (22), denoted by  $g(\alpha)$ , is the following:

$$\begin{aligned} g(\alpha) = & \log\left(M - \left\lceil \frac{1}{\alpha} \right\rceil\right) \\ & + \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor \log \alpha + \left(1 - \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor\right) \log\left(1 - \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor\right) \end{aligned} \quad (23)$$

with the convention that  $0 \log 0$  means 0.

*Proof:* Lets first show that the solution on the right-hand side of (23) forms an upper bound on  $g(\alpha)$ , and then show that this upper bound is tight.

For the derivation of the upper bound, note that due to the above constraints,

$$\begin{aligned} 1 &= \sum_{i=1}^M t_i \stackrel{(a)}{\leq} \alpha |\{i \in \{1, \dots, M\} : t_i > 0\}| \\ &\Rightarrow |\{i \in \{1, \dots, M\} : t_i > 0\}| \geq \frac{1}{\alpha} \\ &\stackrel{(b)}{\Rightarrow} |\{i \in \{1, \dots, M\} : s_i > 0\}| \leq M - \frac{1}{\alpha} \\ &\stackrel{(c)}{\Rightarrow} |\{i \in \{1, \dots, M\} : s_i > 0\}| \leq M - \left\lceil \frac{1}{\alpha} \right\rceil \end{aligned} \quad (24)$$

where inequality (a) holds since  $s_i + t_i \leq \alpha$  and  $s_i, t_i \geq 0$  for every  $i \in \{1, \dots, M\}$ , (b) follows from the constraint that  $s_i t_i = 0$  for every  $i$ , and (c) holds since the cardinality of the support of  $\{s_i\}$  is an integer, and  $\lfloor M - \frac{1}{\alpha} \rfloor = M - \lceil \frac{1}{\alpha} \rceil$ . Hence,

$$-\sum_{i=1}^M s_i \log(s_i) \leq \log\left(M - \left\lceil \frac{1}{\alpha} \right\rceil\right)$$

and the solution of the optimization problem in (22) satisfies

$$g(\alpha) \leq \log\left(M - \left\lceil \frac{1}{\alpha} \right\rceil\right) + f(\alpha) \quad (25)$$

where  $f(\alpha)$  solves the optimization problem

$$\begin{aligned} & \text{maximize} \sum_{i=1}^M t_i \log(t_i) \\ & \text{subject to} \\ & \begin{cases} 0 \leq t_i \leq \alpha, \quad \forall i \in \{1, \dots, M\} \\ \sum_{i=1}^M t_i = 1 \end{cases} \end{aligned} \quad (26)$$

with the  $M$  optimization variables  $t_1, \dots, t_M$ . Note that the objective function in (26) is convex, and the feasible set is a

bounded polyhedron. Furthermore, the maximum of a convex function over a bounded polyhedron is attained at one of its vertices (see, e.g., [39, Corollary 32.3.3]; this property follows from the convex-hull description of a bounded polyhedron and Jensen's inequality). Since the objective function and the feasible set in (26) are invariant to a permutation of the variables  $t_1, \dots, t_M$ , then an optimal point is given by

$$\begin{aligned} t_1 = \dots = t_l = \alpha, \quad l &\triangleq \left\lfloor \frac{1}{\alpha} \right\rfloor \\ t_{l+1} = 1 - \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor, \quad t_{l+2} = \dots = t_M &= 0 \end{aligned}$$

where  $l \leq \frac{M}{2}$  (since  $\alpha \in [\frac{2}{M}, 1]$ ), and indeed  $t_i \in [0, \alpha]$  for  $i \in \{1, \dots, M\}$ . This implies that the solution of the optimization problem in (26) is given by

$$f(\alpha) = \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor \log \alpha + \left(1 - \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor\right) \log\left(1 - \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor\right). \quad (27)$$

From (25) and (27), it follows that the right-hand side of (23) forms an upper bound on  $g(\alpha)$ . It remains to show that this bound is tight. to this end, we separate into the following two cases:

*Case 1:* Suppose that  $N \triangleq \frac{1}{\alpha}$  is an integer. In this case, the upper bound on  $g(\alpha)$  (see the right-hand side of (23)) gets the simplified form

$$g(\alpha) \leq \log\left(M - \frac{1}{\alpha}\right) + \log \alpha = \log(M\alpha - 1).$$

This upper bound on  $g(\alpha)$  is achieved by the point  $(s_1, t_1, \dots, s_M, t_M)$  where

$$\begin{aligned} t_1 = \dots = t_N = \alpha, \quad t_{N+1} = \dots = t_M &= 0 \\ s_1 = \dots = s_N = 0, \quad s_{N+1} = \dots = s_M &= \frac{1}{M - N}. \end{aligned}$$

Note that this point is included in the feasible set of the optimization problem in (22) since  $\frac{1}{M-N} = \frac{\alpha}{M\alpha-1} \leq \alpha$  where the last inequality holds because  $\alpha \in [\frac{2}{M}, 1]$ , and the value of the objective function in (22) at this specific point is equal to

$$\begin{aligned} & -\sum_{i=1}^M s_i \log(s_i) + \sum_{i=1}^M t_i \log(t_i) \\ &= \log\left(M - \frac{1}{\alpha}\right) + \log \alpha = \log(M\alpha - 1) \end{aligned}$$

so this upper bound on  $g(\alpha)$  is tight if  $\frac{1}{\alpha}$  is an integer.

*Case 2:* Suppose that  $\frac{1}{\alpha}$  is not an integer. In this case, let  $l \triangleq \left\lfloor \frac{1}{\alpha} \right\rfloor$  so  $l+1 = \left\lceil \frac{1}{\alpha} \right\rceil$ , and consider the  $(2M)$ -dimensional vector  $(s_1, t_1, \dots, s_M, t_M)$  where

$$\begin{aligned} t_1 = \dots = t_l = \alpha, \quad t_{l+1} = 1 - \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor \\ t_{l+2} = \dots = t_M = 0 \\ s_1 = \dots = s_{l+1} = 0 \\ s_{l+2} = \dots = s_M = \frac{1}{M - l - 1} = \frac{1}{M - \left\lfloor \frac{1}{\alpha} \right\rfloor}. \end{aligned} \quad (28)$$

To verify that it is included in the feasible set of (22), note that due to the constraints of this optimization problem

$$\begin{aligned} 1 &= \sum_{i=1}^M s_i \leq \alpha |\{i \in \{1, \dots, M\} : s_i > 0\}| \\ \Rightarrow |\{i \in \{1, \dots, M\} : s_i > 0\}| &\geq \frac{1}{\alpha} \\ \Rightarrow |\{i \in \{1, \dots, M\} : s_i > 0\}| &\geq \left\lceil \frac{1}{\alpha} \right\rceil \end{aligned}$$

and, by combining it with (24), it follows that

$$\left\lceil \frac{1}{\alpha} \right\rceil \leq |\{i \in \{1, \dots, M\} : s_i > 0\}| \leq M - \left\lceil \frac{1}{\alpha} \right\rceil$$

so  $\left\lceil \frac{1}{\alpha} \right\rceil \leq \frac{M}{2}$ . This implies that for  $j \in \{l+2, \dots, M\}$  (note also that  $\alpha \in [\frac{2}{M}, 1]$ )

$$\begin{aligned} s_j &= \frac{1}{M-l-1} \\ &= \frac{1}{M - \left\lceil \frac{1}{\alpha} \right\rceil} \\ &\leq \frac{2}{M} \\ &\leq \alpha \end{aligned}$$

and  $t_{l+1} = 1 - \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor \leq \alpha$ , so the vector is indeed included in the feasible set of (22). The value of the objective function in (22) at the selected point in (28) is equal to

$$\begin{aligned} &-\sum_{i=1}^M s_i \log(s_i) + \sum_{i=1}^M t_i \log(t_i) \\ &= \log\left(M - \left\lceil \frac{1}{\alpha} \right\rceil\right) \\ &\quad + \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor \log \alpha + \left(1 - \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor\right) \log\left(1 - \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor\right) \\ &= g(\alpha) \end{aligned}$$

so the upper bound on  $g(\alpha)$  from (25) and (27) is tight. This completes the proof of Lemma 1. ■

*Corollary 1:* The solution of the non-convex optimization problem in (22) satisfies the inequality

$$g(\alpha) \leq \log(M\alpha - 1)$$

and this bound is tight if and only if  $\frac{1}{\alpha}$  is an integer.

*Proof:* From Lemma 1 (see Eq. (23)), it follows that

$$\begin{aligned} g(\alpha) &\leq \log\left(M - \frac{1}{\alpha}\right) \\ &\quad + \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor \log \alpha + \left(1 - \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor\right) \log\left(1 - \alpha \left(\frac{1}{\alpha} - 1\right)\right) \\ &= \log\left(M - \frac{1}{\alpha}\right) + \log(\alpha) \\ &= \log(M\alpha - 1) \end{aligned}$$

and the above inequality turns to be an equality if and only if  $\frac{1}{\alpha}$  is an integer. ■

By combining (21) and Corollary 1, it follows that

$$H(V) - H(W) \leq \log(M\alpha - 1)$$

and therefore from (18)

$$\begin{aligned} |H(X) - H(Y)| &\leq d_{\text{TV}}(X, Y) \log(M\alpha - 1) + I(J; \hat{X}) - I(J; \hat{Y}). \end{aligned} \quad (29)$$

Finally, the bound in (11) follows from the inequality

$$I(J; \hat{X}) - I(J; \hat{Y}) \leq H(J) = h(d_{\text{TV}}(X, Y)). \quad (30)$$

We move to derive a refinement of the bound in (11) when  $\frac{1}{2} \leq \frac{P_X}{P_Y} \leq 2$ . In this case, the starting point is the inequality in (29) where it is aimed to improve the upper bound in (30). To this end,

$$\begin{aligned} I(J; \hat{X}) - I(J; \hat{Y}) &= H(J|\hat{Y}) - H(J|\hat{X}) \\ &\leq H(J) - H(J|\hat{X}) \\ &= h(d_{\text{TV}}(X, Y)) - H(J|\hat{X}) \end{aligned} \quad (31)$$

and, from [22, Theorem 11],

$$H(J|\hat{X}) \geq 2 \log 2 \mathbb{P}(J \neq J_{\text{MAP}}(\hat{X})) \quad (32)$$

where  $J_{\text{MAP}}(\hat{X})$  is the maximum a-posteriori (MAP) estimator of  $J$  based on  $\hat{X}$  (note that the minimum on the left-hand side of [22, Eq. (110)] is achieved by the MAP estimator). In the following, the estimator  $J_{\text{MAP}}(\hat{X})$  on the right-hand side of (32) is calculated.

- 1) If  $\hat{X} \notin \text{supp}(P_V)$  then a.s.  $J = 1$  (otherwise,  $J = 0$  and  $\hat{X} = V$ , so  $\hat{X} \in \text{supp}(P_V)$  a.s.). Hence,

$$\hat{X} \notin \text{supp}(P_V) \Rightarrow J_{\text{MAP}}(\hat{X}) = 1.$$

From (5), it follows that  $\hat{X} \notin \text{supp}(P_V)$  if and only if  $P_X(\hat{X}) \leq P_Y(\hat{X})$ .

- 2) If  $\hat{X} \in \text{supp}(P_V)$  then, from (5),  $P_X(\hat{X}) > P_Y(\hat{X})$ . Hence, from (4) and (5) with  $p = 1 - d_{\text{TV}}(X, Y)$ ,

$$\begin{aligned} P_U(\hat{X}) &= \frac{P_Y(\hat{X})}{1 - d_{\text{TV}}(X, Y)} \\ P_V(\hat{X}) &= \frac{P_X(\hat{X}) - P_Y(\hat{X})}{d_{\text{TV}}(X, Y)}. \end{aligned}$$

Since  $U, V, J$  are independent, then from (3)

$$\begin{aligned} \mathbb{P}(J = 1, \hat{X}) &= \mathbb{P}(J = 1) P_U(\hat{X}) = P_Y(\hat{X}) \\ \mathbb{P}(J = 0, \hat{X}) &= \mathbb{P}(J = 0) P_V(\hat{X}) = P_X(\hat{X}) - P_Y(\hat{X}) \end{aligned}$$

so, if  $\hat{X} \in \text{supp}(P_V)$ , then

$$J_{\text{MAP}}(\hat{X}) = \begin{cases} 1 & \text{if } \frac{P_X(\hat{X})}{2} \leq P_Y(\hat{X}) < P_X(\hat{X}) \\ 0 & \text{if } P_Y(\hat{X}) < \frac{P_X(\hat{X})}{2}. \end{cases}$$

To conclude, the MAP estimator of  $J$  that is based on the observation  $\hat{X}$  is given by

$$J_{\text{MAP}}(\hat{X}) = \begin{cases} 1 & \text{if } \frac{P_X(\hat{X})}{2} \leq P_Y(\hat{X}) \\ 0 & \text{if } P_Y(\hat{X}) < \frac{P_X(\hat{X})}{2}. \end{cases}$$

It therefore implies that if  $\frac{P_X}{P_Y} \geq \frac{1}{2}$  whenever  $P_X > 0$ , then  $J_{\text{MAP}}(\hat{X}) = 1$  independently of  $\hat{X}$ , so in this case

$$\mathbb{P}(J \neq J_{\text{MAP}}(\hat{X})) = \mathbb{P}(J = 0) = d_{\text{TV}}(X, Y).$$

Hence, from (31), (32) and the last equality, if  $\frac{P_Y}{P_X} \geq \frac{1}{2}$  whenever  $P_X > 0$  then

$$I(J; \hat{X}) - I(J; \hat{Y}) \leq h(d_{\text{TV}}(X, Y)) - 2 \log 2 \cdot d_{\text{TV}}(X, Y).$$

A combination of the last inequality with (29) finally gives the refined bound in (13). Since it was assumed at the beginning of the proof that  $H(X) \geq H(Y)$  while it is not necessarily known in advance which entropy is larger, the requirement on  $\frac{P_X}{P_Y}$  can be symmetrized by requiring that  $\frac{1}{2} \leq \frac{P_X}{P_Y} \leq 2$  whenever  $P_X, P_Y > 0$ . This completes the proof of Theorem 4. ■

*Corollary 2:* Let  $X$  and  $Y$  be two discrete random variables that take values in a set  $\mathcal{A}$ , and let  $|\mathcal{A}| = M$ . Assume that for some positive constants  $\varepsilon_1, \varepsilon_2$

$$d_{\text{TV}}(X, Y) \leq \varepsilon_1 \leq 1 - \frac{1}{M\varepsilon_2}, \quad (33)$$

$$\frac{d_{\text{loc}}(X, Y)}{d_{\text{TV}}(X, Y)} \leq \varepsilon_2 \leq 1. \quad (34)$$

Then,

$$|H(X) - H(Y)| \leq \varepsilon_1 \log(M\varepsilon_2 - 1) + h(\varepsilon_1). \quad (35)$$

*Proof:* From (11), (12), (34), and since  $\alpha \leq \varepsilon_2$

$$|H(X) - H(Y)| \leq d_{\text{TV}}(X, Y) \log(M\varepsilon_2 - 1) + h(d_{\text{TV}}(X, Y)).$$

The function  $q(\varepsilon) \triangleq \varepsilon c + h(\varepsilon)$  is monotonic increasing over the interval  $[0, \frac{e^c}{1+e^c}]$  ( $q'(\varepsilon) = c + \log(\frac{1-\varepsilon}{\varepsilon}) > 0$  if and only if  $0 < \varepsilon < \frac{e^c}{1+e^c}$ ). Referring to the right-hand side of the above inequality, let  $c \triangleq \log(M\varepsilon_2 - 1)$ , so  $\frac{e^c}{1+e^c} = 1 - \frac{1}{M\varepsilon_2}$ . Hence, if the conditions in (33) and (34) are satisfied then the inequality in (35) holds. ■

*Remark 4:* By considering the pair of probability mass functions  $P_{X,Y}$  and  $P_X \times P_Y$  (without abuse of notation, let  $H(P_X) \triangleq H(X)$ ), then

$$\begin{aligned} H(P_X \times P_Y) - H(P_{X,Y}) \\ &= H(X) + H(Y) - H(X, Y) \\ &= I(X; Y). \end{aligned}$$

Hence, Theorem 4 and Corollary 2 provide bounds on the mutual information between two discrete random variables of finite support, where these bounds are expressed in terms of the local and total variation distances between the joint distribution of  $(X, Y)$  and the product of its marginal distributions. The specialization of Theorem 4 to this setting tightens the bound in [49, Theorem 1], and the former bound is particularized to the latter known bound in the case where the local and total variation distances are equal (which is the extreme case).

*Remark 5:* The bound in [49, Theorem 1] was improved in [34, Proposition 1] without any further assumptions. It is noted that by introducing the additional requirement where there exists some constant  $\varepsilon_2 \in [0, 1]$  such that for every  $y \in \mathcal{Y}$

$$\frac{d_{\text{loc}}(P_X, P_{X|Y=y})}{d_{\text{TV}}(P_X, P_{X|Y=y})} \leq \varepsilon_2$$

then it enables to refine the bound in [34, Proposition 1]. This follows by combining the proof of [34, Proposition 1] with (35) (see Corollary 2) where Eq. (35) replaces the use of [49, Eq. (4)] in [34, Eq. (35)]. The same thing also applies to [35, Proposition 2], referring to its proof in [35, p. 305].

We proceed to consider the entropy difference of discrete random variables in a case of a countable infinite alphabet.

*Theorem 5:* Let  $\mathcal{A} = \{a_1, a_2, \dots\}$  be a countable infinite set. Let  $X$  and  $Y$  be discrete random variables where  $X$  takes values in the set  $\mathcal{X} = \{a_1, \dots, a_m\}$  for some  $m \in \mathbb{N}$ , and  $Y$  takes values in the set  $\mathcal{A}$ . Assume that for some  $\eta_1, \eta_2, \eta_3 > 0$ , the local and total variation distances between  $X$  and  $Y$  satisfy

$$\eta_2 \leq d_{\text{TV}}(X, Y) \leq \eta_1, \quad d_{\text{loc}}(X, Y) \leq \eta_3 \quad (36)$$

where  $\eta_3 \leq \eta_2$ . Let  $M$  be an integer such that

$$\sum_{i=M}^{\infty} P_Y(a_i) \leq \eta_3, \quad M \geq \max\left\{m+1, \frac{\eta_2}{(1-\eta_1)\eta_3}\right\} \quad (37)$$

and let  $\eta_4 > 0$  satisfy

$$-\sum_{i=M}^{\infty} P_Y(a_i) \log P_Y(a_i) \leq \eta_4. \quad (38)$$

Then, the following inequality holds:

$$|H(X) - H(Y)| \leq \eta_1 \log\left(\frac{M\eta_3}{\eta_2} - 1\right) + h(\eta_1) + \eta_4. \quad (39)$$

*Proof:* Let  $\tilde{Y}$  be a random variable that is defined to be equal to  $Y$  if  $Y \in \{a_1, \dots, a_{M-1}\}$ , and it is set to be equal to  $a_M$  if  $Y = a_i$  for some  $i \geq M$ . Hence, the probability mass function of  $\tilde{Y}$  is related to that of  $Y$  as follows:

$$P_{\tilde{Y}}(a_i) = \begin{cases} P_Y(a_i) & \text{if } i \in \{1, \dots, M-1\} \\ \sum_{j=M}^{\infty} P_Y(a_j) & \text{if } i = M. \end{cases} \quad (40)$$

Since  $P_X(a_i) = 0$  for every  $i > m$  and also  $M \geq m+1$  (see the second inequality in (37)), then it follows from (40) that

$$\begin{aligned} d_{\text{TV}}(X, \tilde{Y}) \\ &= \frac{1}{2} \sum_{i=1}^m |P_X(a_i) - P_{\tilde{Y}}(a_i)| + \frac{1}{2} \sum_{i=m+1}^M P_{\tilde{Y}}(a_i) \\ &= \frac{1}{2} \sum_{i=1}^m |P_X(a_i) - P_Y(a_i)| + \frac{1}{2} \sum_{i=m+1}^{\infty} P_Y(a_i) \\ &= d_{\text{TV}}(X, Y). \end{aligned} \quad (41)$$

Hence,  $X$  and  $\tilde{Y}$  are discrete random variables that take values in the set  $\{a_1, \dots, a_M\}$  (note that it includes the set  $\mathcal{X}$ ), and from (36) and (41)

$$0 < \eta_2 \leq d_{\text{TV}}(X, \tilde{Y}) \leq \eta_1. \quad (42)$$

Furthermore, the local distance between  $X$  and  $\tilde{Y}$  satisfies

$$\begin{aligned}
d_{\text{loc}}(X, \tilde{Y}) &= \max_{i \in \{1, \dots, M\}} |P_X(a_i) - P_{\tilde{Y}}(a_i)| \\
&\stackrel{(a)}{=} \max \left\{ \max_{i \in \{1, \dots, M-1\}} |P_X(a_i) - P_Y(a_i)|, \sum_{i=M}^{\infty} P_Y(a_i) \right\} \\
&\stackrel{(b)}{\leq} \max\{d_{\text{loc}}(X, Y), \eta_3\} \\
&\stackrel{(c)}{=} \eta_3
\end{aligned} \tag{43}$$

where (a), (b) and (c) above follow from the equality in (40), the first inequality in (37) and the second inequality in (36), respectively. From (42) and (43)

$$d_{\text{TV}}(X, \tilde{Y}) \leq \eta_1 \triangleq \varepsilon_1 \tag{44}$$

$$\frac{d_{\text{loc}}(X, \tilde{Y})}{d_{\text{TV}}(X, \tilde{Y})} \leq \frac{\eta_3}{\eta_2} \triangleq \varepsilon_2 \tag{45}$$

where  $0 < \varepsilon_2 \leq 1$  (since, by assumption,  $0 < \eta_3 \leq \eta_2$ ). The integer  $M$  is set to satisfy the inequality  $M \geq \frac{\eta_2}{\eta_3(1-\eta_1)}$  (see the second inequality in (37)), so from (44) and (45)

$$\varepsilon_1 \leq 1 - \frac{1}{M\varepsilon_2}.$$

Hence, it follows from Theorem 4 that

$$|H(X) - H(\tilde{Y})| \leq \eta_1 \log\left(\frac{M\eta_3}{\eta_2} - 1\right) + h(\eta_1). \tag{46}$$

Since  $\tilde{Y}$  is a deterministic function of  $Y$  then  $H(Y) \geq H(\tilde{Y})$ , and from (40)

$$\begin{aligned}
|H(\tilde{Y}) - H(Y)| &= H(Y) - H(\tilde{Y}) \\
&= - \sum_{i=M}^{\infty} P_Y(a_i) \log P_Y(a_i) \\
&\quad + \left( \sum_{i=M}^{\infty} P_Y(a_i) \right) \log \left( \sum_{i=M}^{\infty} P_Y(a_i) \right) \\
&\leq - \sum_{i=M}^{\infty} P_Y(a_i) \log P_Y(a_i) \leq \eta_4.
\end{aligned} \tag{47}$$

Finally, the bound in this theorem follows from (46), (47) and the triangle inequality. ■

*Corollary 3:* In the setting of  $X$  and  $Y$  in Theorem 5, assume that  $d_{\text{TV}}(X, Y) \leq \eta$  for some  $\eta \in (0, 1)$ . Let  $M \triangleq \max\left\{m + 1, \frac{1}{1-\eta}\right\}$ , and assume that for some  $\mu > 0$

$$- \sum_{i=M}^{\infty} P_Y(a_i) \log P_Y(a_i) \leq \mu$$

then  $|H(X) - H(Y)| \leq \eta \log(M-1) + h(\eta) + \mu$ .

*Proof:* This corollary follows from Theorem 5 by setting  $\eta_2 = \eta_3 = d_{\text{loc}}(X, Y)$  (note that  $d_{\text{loc}}(X, Y) \leq d_{\text{TV}}(X, Y)$ ), and then  $\eta_1$  and  $\eta_4$  are replaced by  $\eta$  and  $\mu$ , respectively. ■

*Remark 6:* The result in Corollary 3 coincides with [42, Theorem 4], which gives a bound on the entropy difference in terms of the total variation distance by relying on the bound in [49, Eq. (4)] or [21, Theorem 6].

#### IV. EXAMPLES

In the following, we exemplify the use of the new bounds in Section III, and also compare them with some existing bounds.

*Example 1:* Let  $X$  be a discrete random variable that gets values in the set  $\mathcal{A} = \{a_1, \dots, a_M\}$ . Lets express its arbitrary probability mass function in the form

$$P_X(a_i) = \frac{1 + u_i \xi_i}{M} \quad \forall i \in \{1, \dots, M\} \tag{48}$$

where

$$\begin{aligned}
u_i &\in \{-1, 1\}, \quad \xi_i \geq 0, \\
0 &\leq 1 + u_i \xi_i \leq M, \quad \forall i \in \{1, \dots, M\}
\end{aligned}$$

and

$$\sum_{i=1}^M u_i \xi_i = 0$$

where the latter equality is equivalent to  $\sum_{i=1}^M P_X(a_i) = 1$ .

In the following, we derive a lower bound on the entropy  $H(X)$ . Let  $Y$  be a random variable that takes the values from  $\mathcal{A}$  with equal probability, so  $H(Y) = \log M$ . The local and total variation distances between  $X$  and  $Y$  are equal to

$$\begin{aligned}
d_{\text{TV}}(X, Y) &= \frac{1}{2M} \sum_{i=1}^M \xi_i = \frac{\xi_{\text{avg}}^{(M)}}{2} \\
d_{\text{loc}}(X, Y) &= \frac{1}{M} \max_{1 \leq i \leq M} \xi_i = \frac{\xi_{\text{max}}^{(M)}}{M}
\end{aligned}$$

where  $\xi_{\text{avg}}^{(M)}$  and  $\xi_{\text{max}}^{(M)}$  denote the average and maximal values of  $\{\xi_i\}_{i=1}^M$ , respectively. From (12)

$$\alpha_M \triangleq \frac{d_{\text{loc}}(X, Y)}{d_{\text{TV}}(X, Y)} = \frac{2\xi_{\text{max}}^{(M)}}{M\xi_{\text{avg}}^{(M)}}$$

so

$$\alpha_M = \frac{2K_M}{M}$$

where

$$K_M \triangleq \frac{\xi_{\text{max}}^{(M)}}{\xi_{\text{avg}}^{(M)}}. \tag{49}$$

From (11) (where also  $H(Y) = \log M \geq H(X)$ ), it follows that

$$\log M - \frac{\xi_{\text{avg}}^{(M)}}{2} \log(2K_M - 1) - h\left(\frac{\xi_{\text{avg}}^{(M)}}{2}\right) \leq H(X) \leq \log M$$

and, since the binary entropy function is bounded between 0 and  $\log 2$ , the above inequality can be loosened to

$$1 - \frac{\xi_{\text{avg}}^{(M)}}{2} \frac{\log(2K_M - 1)}{\log M} - \frac{\log 2}{\log M} \leq \frac{H(X)}{\log M} \leq 1 \tag{50}$$

which implies (since  $K_M \geq 1$ ) that

$$\lim_{M \rightarrow \infty} \frac{\xi_{\text{avg}}^{(M)} \log K_M}{\log M} = 0 \Rightarrow \lim_{M \rightarrow \infty} \frac{H(X)}{\log M} = 1. \tag{51}$$

For comparison, the bound in Theorem 3 gives that

$$1 - \frac{\xi_{\text{avg}}^{(M)}}{2} \cdot \frac{\log(M-1)}{\log M} - \frac{1}{\log M} \cdot h\left(\frac{\xi_{\text{avg}}^{(M)}}{2}\right) \leq \frac{H(X)}{\log M} \leq 1 \tag{52}$$

which implies that

$$\lim_{M \rightarrow \infty} \xi_{\text{avg}}^{(M)} = 0 \Rightarrow \lim_{M \rightarrow \infty} \frac{H(X)}{\log M} = 1. \quad (53)$$

The latter condition in (53) is strictly stronger than (51). To see this, note that  $1 \leq K_M \leq \frac{M}{2}$  (since  $\frac{2}{M} \leq \frac{d_{\text{loc}}(X,Y)}{d_{\text{TV}}(X,Y)} \leq 1$ ). On the other hand, as a concrete example for the case where the condition in (51) holds whereas the condition in (53) does not hold, let  $M$  be an arbitrary even number, and

$$u_i = (-1)^i, \quad \xi_i = \beta \in (0, 1], \quad i \in \{1, \dots, M\}$$

where, indeed,  $\sum_{i=1}^M u_i \xi_i = \beta \sum_{i=1}^M (-1)^i = 0$ . In this case,  $P_X(a_i) = \frac{1-\beta}{M}$  for odd numbers  $i \in \{1, \dots, M\}$ , and  $P_X(a_i) = \frac{1+\beta}{M}$  for even numbers  $i$ . Furthermore, in this case  $K_M = 1$  for every even  $M$ , so the condition in (51) holds by letting the even number  $M$  tend to infinity. On the other hand, the condition in (53) is not satisfied since  $\lim_{M \rightarrow \infty} \xi_{\text{avg}}^{(M)} = \beta > 0$ . The upper and lower bounds in (52) tend to 1 and  $1 - \frac{\beta}{2}$ , respectively, so the gap between these asymptotic bounds is increased linearly with  $\beta$ . Therefore, Theorem 4 gives a simple lower bound on the entropy  $H(X)$  in terms of the average and maximal values of  $\{\xi_i\}_{i=1}^M$ , which improves the lower bound on the entropy that follows from the known bound in Theorem 3 (see (52)).

For comparison, the bound in [21, Theorem 7] is also applied to this example. In this case, since  $P_X, P_Y \leq \frac{1+\xi_{\text{max}}}{M}$  then  $P_X$  and  $P_Y$  are less than or equal to  $\frac{1}{N_M}$  with  $N_M \triangleq \left\lfloor \frac{M}{1+\xi_{\text{max}}^{(M)}} \right\rfloor$ . Similarly to the above analysis, it is easy to verify from [21, Theorem 7] that

$$\lim_{M \rightarrow \infty} \frac{\xi_{\text{avg}}^{(M)} \log(\xi_{\text{max}}^{(M)})}{\log M} = 0 \Rightarrow \lim_{M \rightarrow \infty} \frac{H(X)}{\log M} = 1. \quad (54)$$

Since

$$\frac{\xi_{\text{avg}}^{(M)} \log(\xi_{\text{max}}^{(M)})}{\log M} \geq \frac{\xi_{\text{avg}}^{(M)} \log(\xi_{\text{avg}}^{(M)})}{\log M} \geq -\frac{\log e}{e \log M}$$

where the right-hand side of this inequality holds since the function  $f(x) = x \log x$  for  $x > 0$  achieves its minimal value at  $x = \frac{1}{e}$ , it follows that if the limit on the left-hand side of (54) is zero then also

$$\lim_{M \rightarrow \infty} \frac{\xi_{\text{avg}}^{(M)} \log(\xi_{\text{avg}}^{(M)})}{\log M} = 0.$$

Therefore, the definition of  $K_M$  in (49) gives that

$$\begin{aligned} & \lim_{M \rightarrow \infty} \frac{\xi_{\text{avg}}^{(M)} \log K_M}{\log M} \\ &= \lim_{M \rightarrow \infty} \frac{\xi_{\text{avg}}^{(M)} \log(\xi_{\text{max}}^{(M)})}{\log M} - \lim_{M \rightarrow \infty} \frac{\xi_{\text{avg}}^{(M)} \log(\xi_{\text{avg}}^{(M)})}{\log M} \\ &= 0. \end{aligned}$$

This shows that the conclusion in (51) implies the one in (54).

*A special case of (48) with numerical results:* As a special case of the probability mass function in (48), let  $M = 2^m$  for some  $m \in \mathbb{N}$ , let  $u_i = (-1)^i$  for  $i \in \{1, \dots, M\}$ , and  $\xi_i = \beta$  for some  $\beta \in [0, 1]$ . In this special case,

$$P_X(a_i) = \begin{cases} 2^{-m}(1-\beta) & \text{if } i \in \{1, 3, \dots, 2^m-1\} \\ 2^{-m}(1+\beta) & \text{if } i \in \{2, 4, \dots, 2^m\}. \end{cases}$$

Let  $Y$  be a random variable that gets all the values in the set  $\{a_1, \dots, a_M\}$  with equal probability (i.e.,  $2^{-m}$ ). Then, the local and total variation distances between  $X$  and  $Y$  are

$$d_{\text{loc}}(X, Y) = \frac{\beta}{M}, \quad d_{\text{TV}}(X, Y) = \frac{\beta}{2}$$

so, from (12),  $\alpha = \frac{2}{M}$ . The entropies of  $X$  and  $Y$  are

$$H(X) = (m-1) \log 2 + h\left(\frac{1-\beta}{2}\right), \quad H(Y) = m \log 2$$

so,  $H(Y) - H(X) = \log 2 - h\left(\frac{1-\beta}{2}\right)$  independently of  $m$ .

For comparison, the known bound in Theorem 3 that only depends on the total variation distance between  $X$  and  $Y$  (with no further knowledge about their probability mass functions) gives

$$H(Y) - H(X) \leq \frac{m\beta}{2} \cdot \log 2 + h\left(\frac{\beta}{2}\right) + \frac{\beta}{2} \cdot \log(1 - 2^{-m})$$

so this upper bound increases almost linearly with  $m$ , in contrast to the exact value that is independent of  $m$ . The new bound in (11), which depends on both the local and total variation distances between  $X$  and  $Y$  (but again, without any further information on their probability mass functions) gives

$$\begin{aligned} & H(Y) - H(X) \\ & \leq d_{\text{TV}}(X, Y) \log(M\alpha - 1) + h(d_{\text{TV}}(X, Y)) \\ & = h\left(\frac{\beta}{2}\right). \end{aligned} \quad (55)$$

Similarly to the exact value, but in contrast to the former bound, the latter bound is independent of  $m$ . Furthermore, if  $\beta \rightarrow 0$  and  $m\beta \rightarrow \infty$ , then the exact value of  $H(Y) - H(X)$  as well as the latter bound (that follows from Theorem 4) tend to zero, whereas the former bound that follows from Theorem 3 tends to infinity. This shows the difference between the two bounds, exemplifying the possible advantage of taking into account the local distance in addition to the total variation distance.

For  $\beta \in [0, \frac{1}{2}]$ , the condition  $\frac{1}{2} \leq \frac{P_X}{P_Y} \leq 2$  is fulfilled, so the tightened bound in (13) gives that

$$0 \leq H(Y) - H(X) \leq h\left(\frac{\beta}{2}\right) - \beta \log 2. \quad (56)$$

If  $\beta = \frac{1}{2}$ ,  $H(Y) - H(X) = \log 2 - h\left(\frac{1}{2}\right) = 0.131$  nats, the upper bound in (55) is equal to 0.562 nats, and the tightened version of this bound in (56) is equal to 0.216 nats.

It is noted that since  $P_X$  is majorized by  $P_Y$  (see [22, Definition 1 on p. 5934]), then according to [22, Theorem 3]

$$H(Y) - H(X) \geq D(P_X || P_Y)$$

and since  $P_Y$  refers to a uniform distribution over a set of cardinality  $M = 2^m$  then  $H(Y) = m \log 2$ , and

$$D(P_X || P_Y) = m \log 2 - H(P_X)$$

so, the above lower bound is achieved here with equality.

In Example 1, the probability mass function of the discrete random variable  $X$  was known explicitly. However, in many interesting applications, this is not necessarily the case. If the exact distribution of  $X$  is not available or is numerically



hard to compute, a derivation of some good bounds on the local and total variation distances between  $X$  and another random variable  $Y$  with a known probability mass function can be valuable to get a rigorous bound on the difference  $|H(X) - H(Y)|$  via Theorems 4 or 5. As a result of the calculation of such a bound on the entropy difference, it provides bounds on the entropy of  $X$  in terms of another entropy (the entropy of  $Y$ ) which is assumed to be easily calculable. For example, assume that  $X = \sum_{i=1}^n X_i$  is expressed as a sum of Bernoulli random variables that are either independent or weakly dependent, and may be also non-identically distributed. Let  $X_i \sim \text{Bernoulli}(p_i)$ , and assume that  $\sum_{i=1}^n p_i = \lambda$  where all of the  $p_i$ 's are much smaller than 1. In this case, the approximation of  $X$  by a Poisson distribution with mean  $\lambda$  (according to the law of small numbers [24]) raises the question: How close is  $H(X)$  to the entropy of the Poisson distribution with mean  $\lambda$ ? (note that the latter entropy of the Poisson distribution is calculated efficiently in [2]). This question is especially interesting because the support of the Poisson distribution is the infinite countable set of non-negative integers, and the entropy is known not to be continuous when the support is not finite; hence, a small total variation distance does not in general yield a small difference between the two entropies. This question was addressed in [42, Section 2] via the use of Corollary 3 (which coincides with [42, Theorem 4]), combined with an upper bound on the total variation distance between  $X$  and  $Y$  where the latter bound is calculated via the use of the Chen-Stein method (see, e.g., [40, Chapter 2]).

*Example 2:* In the following, we wish to tighten the bounds on the entropy of a sum of independent Bernoulli random variables that are not necessarily identically distributed. The bound provided in [42, Proposition 1] relies on an upper bound on the total variation distance between this sum and a Poisson random variable with the same mean (see [4, Theorem 1] or [5, Theorem 2.M]). In order to tighten the bound on the entropy in the considered setting, we further rely on a lower bound on the total variation distance (see [42, Theorem 6 and Corollary 2]) and an upper bound on the local distance (see [5, Theorem 2.Q and Corollary 9.A.2]). The latter two bounds provide an upper bound on the ratio of the local and total variation distances, which enables to apply the bound in Theorem 5; it improves the bound in Corollary 3 which solely relies on an upper bound on the total variation distance. It is noted that the latter looser bound, which relies on Corollary 3 (or [42, Theorem 4]) was used in [42, Section 2] for estimating the entropy of a sum of Bernoulli random variables in the more general setting where the summands are possibly dependent.

Let  $X = \sum_{i=1}^n X_i$  be a sum of independent Bernoulli random variables with  $X_i \sim \text{Bernoulli}(p_i)$  for  $i \in \{1, \dots, n\}$ . Let  $\sum_{i=1}^n p_i = \lambda$ , and let  $Y \sim \text{Po}(\lambda)$  be a Poisson random variable with mean  $\lambda$ . From [4, Theorem 1] (or [5, Theorem 2.M]), the following upper bound on the total variation distance holds:

$$d_{\text{TV}}(X, Y) \leq \left( \frac{1 - e^{-\lambda}}{\lambda} \right) \sum_{i=1}^n p_i^2. \quad (57)$$

Furthermore, from [42, Corollary 2], the following lower

bound on the total variation distance holds:

$$d_{\text{TV}}(X, Y) \geq k \sum_{i=1}^n p_i^2 \quad (58)$$

where

$$k \triangleq \frac{e}{2\lambda} \frac{1 - \frac{1}{\theta} \left(3 + \frac{7}{\lambda}\right)}{\theta + 2e^{-1/2}} \quad (59)$$

$$\theta \triangleq 3 + \frac{7}{\lambda} + \frac{1}{\lambda} \cdot \sqrt{(3\lambda + 7)[(3 + 2e^{-1/2})\lambda + 7]}. \quad (60)$$

An upper bound on the local distance between a sum of independent Bernoulli random variables and a Poisson distribution with the same mean  $\lambda$  follows as a special case of [5, Corollary 9.A.2] by setting  $l = 1$  (so that the distribution  $Q_l$  in this corollary is specialized for  $l = 1$  to the Poisson distribution  $\text{Po}(\lambda)$ , according to [5, Eq. (1.12) on p. 177]). Since the upper bound on the right-hand side of the inequality in [5, Corollary 9.A.2] does not depend on the (time) index  $j$ , it follows that the same bound also holds while referring to  $d_{\text{loc}}(X, Y) \triangleq \sup_{j \in \mathbb{N}_0} |\mathbb{P}(X = j) - \text{Po}(\lambda)\{j\}|$ . Based on the notation used in this corollary, it implies that if  $\left(\frac{1 - e^{-\lambda}}{\lambda}\right) \sum_{i=1}^n p_i^2 \leq \frac{1}{8}$  then the local distance between a sum of independent Bernoulli random variables  $X_i \sim \text{Bernoulli}(p_i)$  and a Poisson random variable with mean  $\lambda = \sum_{i=1}^n p_i$  is upper bounded by

$$\begin{aligned} d_{\text{loc}}(X, Y) &\leq 4 \left( 2 \max_{j \in \mathbb{N}_0} \mathbb{P}(Y = j) \right) \left( \frac{1 - e^{-\lambda}}{\lambda} \right) \sum_{i=1}^n p_i^2 \\ &\stackrel{(a)}{\leq} 4 \min \left\{ \sqrt{\frac{2}{e\lambda}}, 2e^{-\lambda} I_0(\lambda) \right\} \left( \frac{1 - e^{-\lambda}}{\lambda} \right) \sum_{i=1}^n p_i^2 \quad (61) \end{aligned}$$

where inequality (a) holds due to [5, Proposition A.2.7 on pp. 262–263], and  $I_0$  denotes the modified Bessel function of order zero. Since an upper bound on the total variation distance also forms an upper bound on the local distance, then a combination of (57) and (61) gives that

$$\begin{aligned} d_{\text{loc}}(X, Y) &\leq \min \left\{ 1, 4 \sqrt{\frac{2}{e\lambda}}, 8e^{-\lambda} I_0(\lambda) \right\} \left( \frac{1 - e^{-\lambda}}{\lambda} \right) \sum_{i=1}^n p_i^2. \quad (62) \end{aligned}$$

We now apply Theorem 5 to get rigorous bounds on the entropy  $H(X)$  by estimating how close it is to  $H(\text{Po}(\lambda))$ . Note that the improvement in the tightness of the bound in Theorem 5, in comparison to the looser bound in Corollary 3, is more remarkable when the ratio  $\alpha$  of the local and total variation distances is close to zero. This happens to be the case if  $\lambda \gg 1$  where due to the asymptotic expansion of  $I_0$  (see [1, Eq. (9.7.1) on p. 377] or [15, Eq. (8.451.5) on p. 973])

$$I_0(\lambda) \approx \frac{e^\lambda}{\sqrt{2\pi\lambda}} \left( 1 + \frac{1}{8\lambda} + \frac{9}{128\lambda^2} + \dots \right), \quad \text{if } \lambda \gg 1$$

one gets from Eqs. (58)–(60) and (62), combined with the

limit in [42, Eq. (149)], that

$$\begin{aligned} \alpha &= \frac{d_{\text{loc}}(X, Y)}{d_{\text{TV}}(X, Y)} \\ &\stackrel{(\text{if } \lambda \gg 1)}{\leq} \frac{4\sqrt{\frac{2}{\pi\lambda}} \left(\frac{1-e^{-\lambda}}{\lambda}\right) \sum_{i=1}^n p_i^2}{\frac{e}{6} \left(1 + \sqrt{1 + \frac{2}{3} \cdot e^{-1/2}}\right)^{-2} \left(\frac{1-e^{-\lambda}}{\lambda}\right) \sum_{i=1}^n p_i^2} \\ &= \frac{24}{e} \sqrt{\frac{2}{\pi}} \left(1 + \sqrt{1 + \frac{2}{3} \cdot e^{-1/2}}\right)^2 \sqrt{\frac{1}{\lambda}} \\ &\approx \frac{33.634}{\sqrt{\lambda}} \end{aligned} \quad (63)$$

so, for large values of  $\lambda$ , the upper bound on the parameter  $\alpha$  in (12) decays to zero like the square-root of  $\frac{1}{\lambda}$ .

As a possible application, consider a noiseless binary-adder multiple-access channel (MAC) with  $n$  independent users where each user transmits binary symbols, and the channel output is the algebraic sum of the input symbols. The capacity region of this MAC channel is an  $n$ -dimensional polyhedron. One feature of this capacity region is the sum of the rates that is given by  $R_{\text{SUM}} \triangleq \sum_{i=1}^n R_i$ , and it is upper bounded by the joint mutual information between the input symbols  $X_1, \dots, X_n$  and the corresponding channel output  $Y = \sum_{i=1}^n X_i$ , i.e.,

$$R_{\text{SUM}} \leq \max_{P_{\mathbf{X}}: P_{\mathbf{X}}=P_{X_1} \dots P_{X_n}} I(X_1, \dots, X_n; Y)$$

where, since the MAC is noiseless and the output symbol is the sum of the  $n$  input symbols then  $H(Y|X_1, \dots, X_n) = 0$ , and therefore  $I(X_1, \dots, X_n; Y) = H(Y)$ .<sup>1</sup> Hence, in the considered setting, the maximal sum rate is the maximal entropy of the sum of  $n$  independent binary random variables where  $X_i \sim \text{Bernoulli}(p_i)$  for  $i \in \{1, \dots, n\}$ . Under the constraint that  $\sum_{i=1}^n \mathbb{E}[X_i] \leq \lambda$ , it follows from the maximal entropy result in [17], [23] and [43] that the entropy of  $Y$  is maximized when the  $n$  independent inputs are i.i.d. with mean  $p = \frac{\lambda}{n}$ , and consequently the channel output  $Y$  is Binomially distributed with  $Y \sim \text{Binom}(n, \frac{\lambda}{n})$ . For a very large number of users, the calculation of the entropy of the Binomial distribution is difficult, and it would be much easier to calculate the entropy  $H(\text{Po}(\lambda))$  for a Poisson distribution with mean  $\lambda$  (see [2]).

In the following, we make use of Theorem 5 to get an upper bound on the entropy difference

$$H(\text{Po}(\lambda)) - H\left(\text{Binom}\left(n, \frac{\lambda}{n}\right)\right) \quad (64)$$

where, due to the maximal entropy result for the Poisson distribution (see, e.g., [17], [23] or [43]), this difference is positive. Let  $X \sim \text{Binom}(n, \frac{\lambda}{n})$  be a sum of  $n$  i.i.d. Bernoulli random variables with probability of success  $p = \frac{\lambda}{n}$ , and let  $Y \sim \text{Po}(\lambda)$ . From (57), the total variation distance in this case

is upper bounded by

$$d_{\text{TV}}(X, Y) \leq \frac{\lambda(1 - e^{-\lambda})}{n} \triangleq \eta_1. \quad (65)$$

From (58) and (59), the following inequality holds:

$$d_{\text{TV}}(X, Y) \geq \frac{e}{2} \frac{1 - \frac{1}{\theta} \left(3 + \frac{7}{\lambda}\right)}{\theta + 2e^{-1/2}} \frac{\lambda}{n} \triangleq \eta_2 \quad (66)$$

where  $\theta$  is given in (60). Furthermore, for using Theorem 5, one needs an upper bound on the local distance between the Poisson and Binomial distributions. Eq. (62) gives that

$$\begin{aligned} d_{\text{loc}}(X, Y) &\leq \min \left\{ 1, 4\sqrt{\frac{2}{\pi\lambda}}, 8e^{-\lambda} I_0(\lambda) \right\} \frac{\lambda(1 - e^{-\lambda})}{n} \triangleq \eta_3. \end{aligned} \quad (67)$$

Following the notation in Theorem 5, it follows that  $m = n+1$ . From (37), one needs to choose an integer  $M$  such that

$$M \geq \max \left\{ n+2, \frac{\eta_2}{\eta_3(1 - \eta_1)} \right\} \quad (68)$$

and

$$\sum_{j=M}^{\infty} \Pi_{\lambda}(j) \leq \eta_3 \quad (69)$$

where  $\Pi_{\lambda}(j) \triangleq \frac{e^{-\lambda} \lambda^j}{j!}$  for  $j \in \mathbb{N}_0$  designates the probability mass function of  $\text{Po}(\lambda)$ . Based on Chernoff's inequality,

$$\begin{aligned} \sum_{j=M}^{\infty} \Pi_{\lambda}(j) &= \mathbb{P}(Y \geq M) \\ &\leq \exp \left\{ - \left[ \lambda + M \ln \left( \frac{M}{\lambda e} \right) \right] \right\}. \end{aligned} \quad (70)$$

Let  $M \geq \lambda e^2$ , then it follows from (69) and (70) that it is sufficient for  $M$  to satisfy the condition  $\exp(-(\lambda + M)) \leq \eta_3$ . Combining it with (68) leads to the following possible choice of  $M$ :

$$M \triangleq \max \left\{ n+2, \frac{\eta_2}{\eta_3(1 - \eta_1)}, \lambda e^2, \ln \left( \frac{1}{\eta_3} \right) - \lambda \right\} \quad (71)$$

where  $\eta_1$ ,  $\eta_2$  and  $\eta_3$  are introduced in (65), (66), and (67) respectively. Finally, for the use of Theorem 5, one needs to choose  $\eta_4 > 0$  such that  $\sum_{j=M}^{\infty} \{-\Pi_{\lambda}(j) \log(\Pi_{\lambda}(j))\} \leq \eta_4$ . From the analysis in [42, Eqs. (43)–(47)], it follows from the last inequality and [42, Eq. (47)] that  $\eta_4$  here is equal to  $\mu$  in [42, Eq. (23)], i.e.,

$$\begin{aligned} \eta_4 &\triangleq \left[ \left( \lambda \log \left( \frac{e}{\lambda} \right) \right)_+ + \lambda^2 + \frac{6 \log(2\pi) + 1}{12} \right] \\ &\quad \exp \left\{ - \left[ \lambda + (M-2) \log \left( \frac{M-2}{\lambda e} \right) \right] \right\} \end{aligned} \quad (72)$$

where  $M$  is introduced in (71), and  $(x)_+ \triangleq \max\{x, 0\}$  for every  $x \in \mathbb{R}$ . At this stage, we are ready to apply Theorem 5 to derive a bound on the non-negative difference between the entropies in (64). From Theorem 5, it follows that

$$\begin{aligned} 0 &\leq H(\text{Po}(\lambda)) - H\left(\text{Binom}\left(n, \frac{\lambda}{n}\right)\right) \\ &\leq \eta_1 \log \left( \frac{M\eta_3}{\eta_2} - 1 \right) + h(\eta_1) + \eta_4. \end{aligned} \quad (73)$$

<sup>1</sup>The reader is referred to [6] for the consideration of the sum-rate for two noiseless multiple-access channels with some similarity to the binary adder channel, see footnote in [6, p. 43].

For comparison, it follows from Corollary 3 that the upper bound on the right-hand side of (73) is replaced by

$$\eta_1 \log(\tilde{M} - 1) + h(\eta_1) + \eta_4 \quad (74)$$

where

$$\tilde{M} \triangleq \max\left\{n + 2, \frac{1}{1 - \eta_1}\right\}. \quad (75)$$

Note that the bound in (73) improves the bound in (74) if  $\eta_3 < \eta_2$  (i.e., if the upper bound on the local distance is smaller than the lower bound on the total variation distance). Furthermore, the latter bound does not take into account the parameters  $\eta_2$  and  $\eta_3$ . As a numerical example, for  $n = 10^6$  and  $p = 0.1$ , let's check the bound on the entropy difference in (64) for  $\lambda = np$  (i.e.,  $\lambda = 10^5$ ). Eqs. (65)–(67), (71), (72) and (75) yield that

$$\eta_1 = 10^{-1}, \eta_2 = 9.5 \cdot 10^{-3}, \eta_3 = 1.0 \cdot 10^{-3}, \eta_4 \approx 0, \\ M = \tilde{M} = 10^6 + 2$$

and the two bounds in (73) and (74) are, respectively, equal to 1.483 and 1.707 nats, respectively. The value of  $H(\text{Po}(\lambda))$  is 7.175 nats, so the entropy  $H(\text{Binom}(n, \frac{\lambda}{n}))$  ranges between 5.693 to 7.175 nats. Note that for  $n = 10^6$  and  $\lambda = 10^4$ , where  $p = \frac{\lambda}{n}$  is decreased from  $10^{-1}$  to  $10^{-2}$ , the upper bounds on (64) are decreased, respectively, to 0.183 and 0.194 nats, and  $H(\text{Po}(\lambda)) = 6.024$  nats. The Poisson approximation is more accurate in the latter case, consistently with the law of small numbers (see, e.g., [24]).

*Remark 7:* Example 2 considers the use of Theorem 5 for the estimation of the entropy of a sum of independent Bernoulli random variables. The more general case of the estimation of the entropy (via rigorous bounds) for a sum of possibly dependent Bernoulli random variables was considered in [42, Section II] by using the looser bound in Corollary 3 with an upper bound on the total variation distance that follows from the Chen-Stein method (see [3, Theorem 1]). It is noted that, in principle, also the sharper bound in Theorem 5 can be applied to obtain bounds on the entropy for a sum of possibly dependent Bernoulli random variables. To this end, in addition to the upper bound on the total variation distance in [3, Theorem 1], one needs to rely on a lower bound on the total variation distance (see [5, Chapter 3]) and an upper bound on the local distance (see [5, Theorem 2.Q on p. 42]). It is noted, however, that these distance bounds are much simplified in the setting of independent summands (see Example 2).

*Remark 8:* The Chen-Stein method for the Poisson approximation was adapted in [28] to the setting of the geometric distribution, and it yields a convenient method for assessing the accuracy of the geometric approximation to the distribution of the number of failures preceding the first success in dependent trials. A recent study of upper bounds on the total variation and local distances for the geometric approximation (respectively, denoted by  $d_1$  and  $d_2$  in [29]) enables to apply the entropy bounds in Theorem 5 and Corollary 3 in a conceptually similar way to Example 2. Furthermore, the entropy bound in Corollary 3 can be applied to compound geometric and negative binomial approximations, based on upper bounds on the total variation distance that were derived via Stein's method in [12] and [45], respectively.

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